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A CLASS OF BACKWARD FREE-CONVECTIVE BOUNDARY-LAYER SIMILARITY SOLUTIONS

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Abstract—This paper presents a class of backward free-convective boundary-layer similarity solutions. It is shown that these boundary layers can be produced along slender downward-projecting slabs of prescribed thickness variation, which are infinitely long. It is pointed out that these solutions can be used to describe free-convective flows along vertical fins which have received attention in the literature before. An important result is that the temperature along a downward-projecting fin of constant thickness varies in proportion to the inverse of the seventh power of the shifted longitudinal coordinate.

NOMENCLATURE

f,	similarity function, equation (13);
g,	acceleration due to gravity [m s^{-2}];
Gr_x,	local Grashof number, equation (12);
H,	dimensionless heat-transfer group, equation (31);
k,	thermal conductivity of fluid [W m^{-1} K^{-1}];
k_s,	thermal conductivity of slab material [W m^{-1} K^{-1}];
L,	length of fin used in ref. [1] [m];
m,	exponent defining slab-thickness variation, equation (1);
n,	exponent defining slab-temperature variation, equation (11);
$\bar{n}$,	exponent defining temperature variation, equation (29);
N_{ce},	parameter defined in ref. [1];
q,	local wall heat flux [W m^{-2}];
Q,	heat flux at the top of the slab [W m^{-2}];
$R(N_{ce})$,	function defined by equation (33);
T,	temperature [K];
T_w,	temperature at the wall [K];
T_∞,	ambient temperature [K];
ΔT,	maximum temperature difference [K];
u,	velocity component in the x direction [m s^{-1}];
v,	velocity component in the y direction [m s^{-1}];
x,	longitudinal coordinate [m];
x_0,	shift parameter [m];
y,	normal coordinate [m].

Greek symbols

β,	coefficient of thermal expansion [K^{-1}];
δ,	thickness of the slab [m];
δ_0,	slab thickness at $x = 0$ [m];
η,	similarity variable, equation (14);
θ,	dimensionless temperature, equation (13);
κ,	thermal diffusivity [m^2 s^{-1}];
ν,	kinematic viscosity [m^2 s^{-1}];
σ,	Prandtl number, ν/κ;
ψ,	stream function [m^2 s^{-1}].

1. INTRODUCTION

IN A RECENT paper Sparrow and Acharya [1] considered the problem of free convection past a hot downward-projecting fin of constant thickness. The basic motivation for their study was to present an improvement upon older treatments in this field which assume a uniform heat-flux distribution along the fin. Sparrow and Acharya were right in remarking that the heat flux cannot be prescribed, but should rather be the outcome of an analysis of the relevant free-convection problem.

Their solution was of the finite-difference type, as they were unable to derive a similarity solution for a fin of uniform thickness. Indeed, such a similarity solution does not exist in the traditional sense with the boundary layer originating at the leading edge which coincides with the lower end of the fin. A decade or so before, Lock and Gunn [2] tackled a similar problem. These authors considered fins of variable thickness. The thickness variation was chosen in such a way that a similarity solution belonging to the Sparrow–Gregg–Finston [3, 4] class could be employed. These fins were thick at the top and tapered down to a cusped leading edge. As such, their analysis was rather of theoretical than of practical interest.

The purpose of the present work is to show that flows along hot downward-projecting fins of uniform thickness can be modelled by similarity solutions. However, the boundary-layer similarity solutions are of a kind for which Goldstein coined the term "backward" [5]. These are boundary layers which have lost any memory of phenomena and conditions prevalent near the leading edge. It would seem as if the leading edge had receded to stations infinitely far upstream. Therefore, in the solutions to these problems any reference to a leading edge is absent. The dependence of the longitudinal coordinate involves a shift parameter that is determined by an additional condition such as the location of a sink or the prescription of a temperature at a certain location.

In a recent paper the present author [6] introduced the concept of a backward boundary layer in free convection. That paper concerned the flow along a vertical channel in which hot fluid was flowing in the downward direction. The channel extended infinitely far in the downward direction, its wall being insulated above a given location. The present paper sets out to show that the similarity solution given in ref. [6], and that along a fin of uniform thickness, belong to a wider class of backward free-convective boundary-layer similarity solutions. This class of solutions is, so to say, the counterpart of the family of similarity solutions derived by Sparrow and Gregg [3] and independently by Finston [4] for classical forward flows.

2. STATEMENT OF THE PROBLEM

To produce the said class of backward free-convective boundary layers we shall consider a semi-infinite slab of varying thickness (Fig. 1). To ensure a 2-dim. flow, the slab is assumed to be infinitely wide in both directions normal to the plane of the drawing. One side of the slab is flat. Using cartesian coordinates x and y, we shall denote this face by $y = 0$. The coordinate x measures distance along this plane, the direction of positive x being parallel but opposite to the acceleration due to gravity g. The top end of the slab being located at $x = 0$, all portions of the slab are seen to be in the domain of negative x. The slab is assumed to be hotter than its surroundings. Therefore, the flow in the adjacent fluid will be upward.

The other side of the slab, which is insulated, generally is curved, and its position is given by

$$y = -\delta(x) = -\delta_0\left(1-\frac{x}{x_0}\right)^{-m}. \tag{1}$$

It should be noted that the singularity at $x = x_0 > 0$ is outside our field of interest, as we shall be concerned with the flow along the slab only, not above it. Any singularities in the solution will be at $x = x_0$ and are therefore devoid of any meaning. In ref. [6], the parameter x_0, which is a kind of shift parameter,

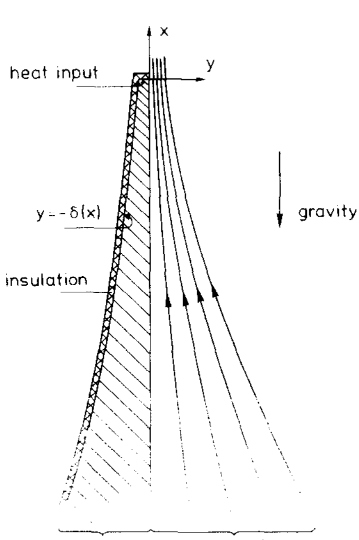

FIG. 1. Geometry. Case depicted refers to $-3/2 < m < -1$.

appeared as the result of a requirement on the conservation of energy. We refer to that paper for a further discussion on this matter.

Although at the moment we do not have any means of knowing which values of the exponent m are admissible, we shall see later that there is a lower limit of m below which no solution can be found. The thickness parameter δ_0 is assumed to be small, i.e. the slab is thought to be slender. A more precise way of expressing this would be that $\delta_0 \ll |x|$ in the thicker part of the boundary layer. The temperature distribution in the slab is taken as 1-dim., i.e. $T \sim T_w(x)$ within the slab, where $T_w(x)$ is the temperature in the plane $y = 0$. As was argued in [2], this assumption leads to a simple ordinary differential equation for T_w

$$\frac{d}{dx}\delta\frac{dT_w}{dx} + \frac{k}{k_s}\frac{\partial T}{\partial y}\bigg|_{y=+0} = 0 \tag{2}$$

where k is the thermal conductivity of the fluid surrounding the slab and k_s is that of the slab material. It should be understood that the normal derivative in equation (2) is taken at the fluid-side of the face at $y = 0$. Later on we shall derive a precise condition for the validity of the 1-dim. assumption.

The slab is heated by a continual heat input Q at $x = 0$, coming from some extraneous source. In an actual experiment one might accomplish this by electrical resistance heating. The heat will be conducted into the slab, a process which is expressed by the first term of equation (2). Part of this heat is given off to the surrounding fluid, which finds its expression in the second term of equation (2). More and more of the excess heat leaves the slab as we go downwards, until all of it will be in the fluid. Sufficiently far downwards, the temperature of the slab will be equal to that of the far ambient, i.e.

$$\lim_{x\to-\infty} T_w = T_\infty \tag{3}$$

where T_∞ is the lowest temperature in the system. Clearly, we should have

$$Q = -k\int_{-\infty}^{0}\frac{\partial T}{\partial y}\bigg|_{y=+0} dx. \tag{4}$$

If the Grashof number is sufficiently larger than unity, the heating of the surrounding fluid will lead to a boundary-layer flow, described in the usual way by the set of equations

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{5}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} + g\beta(T-T_\infty), \tag{6}$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \kappa\frac{\partial^2 T}{\partial y^2} \tag{7}$$

where u and v are the velocity components in the x and y directions, respectively. Further we have: ν, kinematic viscosity; κ, thermal diffusivity; β, coefficient of thermal expansion. The boundary conditions are

$$u = 0, \quad v = 0, \quad T = T_w(x) \quad \text{at } y = 0, \tag{8}$$

$$u \to 0, \quad T \to T_\infty, \quad \text{if } y \to \infty. \tag{9}$$

In addition to equation (3) we require

$$u \to 0 \quad \text{if } x \to -\infty. \tag{10}$$

The governing equations being parabolic, we do not need more conditions. Indeed, what happens in a plane denoted by a certain value of x is completely determined by conditions prevailing at lower, i.e. more negative, values of x. Therefore, conditions given at $y = 0$, $y \to \infty$ and $x \to -\infty$ will suffice.

3. SIMILARITY ANALYSIS

If the present system is to have a similarity solution, the functional description of the temperature variation in the x-direction ought to be in accordance with that of $\delta(x)$ defined by equation (1), i.e.

$$T_w(x) = T_\infty + \Delta T\left(1 - \frac{x}{x_0}\right)^{-n} \tag{11}$$

where ΔT is a temperature difference that is still to be determined. At this stage the exponent n is unknown in relation to m. Defining the local Grashof number Gr_x as follows:

$$Gr_x = \frac{g\beta[T_w(x) - T_\infty](x_0 - x)^3}{\nu^2} \tag{12}$$

and employing the usual definition of the stream function ψ, with $u = \partial\psi/\partial y$ and $v = -\partial\psi/\partial x$, we introduce the transformation

$$\psi = \nu Gr_x^{1/4} f(\eta); \quad T = T_\infty + \Delta T\left(1 - \frac{x}{x_0}\right)^{-n}\theta(\eta) \tag{13}$$

$$\eta = \frac{y}{x_0 - x} Gr_x^{1/4}, \tag{14}$$

which is modelled on the one introduced by Sparrow and Gregg [3] and Finston [4]. However, we emphasize that the parameter range of n will be completely different from the one considered by these authors. In fact it will be complementary.

The set of equations (5)–(7) is now reduced to

$$f''' + \frac{n-3}{4} ff'' + \frac{1-n}{2}(f')^2 + \theta = 0, \tag{15}$$

$$\sigma^{-1}\theta'' + \frac{n-3}{4} f\theta' - nf'\theta = 0 \tag{16}$$

where $\sigma = \nu/\kappa$ is the Prandtl number and a prime denotes differentiation with respect to argument. The boundary conditions read

$$\eta = 0: \quad f = f' = 0, \quad \theta = 1, \tag{17}$$

$$\eta \to \infty: \quad f' \to 0, \quad \theta \to 0. \tag{18}$$

Inspection of equations (15) and (16) reveals that for slowly decaying temperature distributions along the wall, i.e. for values of n in the interval $0 < n < 3$, the coefficients of the second term are negative. In free convection such cases have not been considered to date.

In order to determine the compatibility of the formal solution given by equations (13) and (14) with the temperature field in the slab, we substitute equations (11) and (13) in (2). As a result of this action we obtain the compatibility conditions

$$m = \frac{n-7}{4} \tag{19}$$

and

$$\delta_0 = \frac{4x_0}{n(5n-3)}\left(\frac{k}{k_s}\right)[-\theta'(0)]\, Gr_0^{1/4}. \tag{20}$$

In principle, the first of these was already given by Lock and Gunn [2]. The second condition prescribes the value of ΔT through Gr_0. The required heat flux at $x = 0$ is then given by

$$Q = nk_s\Delta T/x_0. \tag{21}$$

4. NUMERICAL SOLUTION

In this section we shall present a concise description of the methods that are most suitable for the solution of the system equations (15)–(18). Referring to ref. [7], we may state that the asymptotic behaviour of the solution for $\eta \to \infty$ strongly depends upon the signs of the second terms in the equations (15) and (16). If these are non-positive, the behaviour will be algebraic, otherwise it will be exponential. If $n \leqslant 3$ we may write

$$f \sim \bar{\eta}^{\alpha} \sum_{i=0}^{N} a_i \bar{\eta}^{-i\bar{\alpha}}; \quad \theta \sim \bar{\eta}^{\beta} \sum_{i=0}^{N} b_i \bar{\eta}^{-i\bar{\alpha}}, \tag{22}$$

if $\eta \to \infty$, with

$$\bar{\eta} = \eta - \eta_0, \quad \alpha = \frac{3-n}{1+n}, \quad \beta = -\frac{4n}{1+n}, \quad \bar{\alpha} = 1 + \alpha \tag{23}$$

where η_0 is an unknown constant that will be determined through numerical integration. As the expansions of equation (22) are usually of a divergent asymptotic nature, the value of N should be chosen so as to give the smallest truncation errors.

Substituting the asymptotic series in equations (15) and (16) we may derive the following recurrence relations for the coefficients a_i and b_i, where $i \gg 1$:

$$\begin{aligned} a_i = &-\left\{\frac{n-3}{4}[\alpha(\alpha-1) + (\alpha - i\bar{\alpha})(\alpha - 1 - i\bar{\alpha})] \right.\\ &\left. + (1-n)\alpha(\alpha - i\bar{\alpha})\right\}^{-1} a_0^{-1} \\ &\times \left\{[\alpha - (i-1)\bar{\alpha}][\alpha - 1 - (i-1)\bar{\alpha}]\right. \\ &\times [\alpha - 2 - (i-1)\bar{\alpha}]a_{i-1} + b_{i-1} \\ &+ \sum_{j=1}^{i-1}\left[\frac{n-3}{4}(\alpha - 1 - j\bar{\alpha}) + \frac{1-n}{2}(\alpha - (i-j)\bar{\alpha})\right] \\ &\left.\times (\alpha - j\bar{\alpha})a_j a_{i-j}\right\}, \end{aligned} \tag{24}$$

$$b_i = -\left[\frac{n-3}{4}(\beta - i\bar{\alpha}) - n\alpha\right]^{-1} a_0^{-1}$$
$$\times \left\{\sigma^{-1}[\beta-(i-1)\bar{\alpha}][\beta-1-(i-1)\bar{\alpha}]b_{i-1} + \sum_{j=0}^{i-1}\left[\frac{n-3}{4}(\beta - j\bar{\alpha}) - n(\alpha-(i-j)\bar{\alpha})\right]a_{i-j}b_j\right\}. \quad (25)$$

The coefficients a_0 and b_0 are as yet undetermined. Through the parameters a_0, b_0 and η_0 the general solution has three degrees of freedom. These are necessary to satisfy the three boundary conditions at $\eta = 0$ given by equation (17). The boundary conditions for $\eta \to \infty$ are satisfied if $\alpha < 1$ and $\beta < 0$. This means that $n > 1$.

As in our previous work [6, 7], it proves to be impracticable to solve the system of equations (15) and (16) in the traditional way by a simple forward integration procedure starting at $\eta = 0$, i.e. when $1 < n \leqslant 3$. We have to use equation (22) and select the correct values of a_0, b_0 and η_0. A very economical way of doing so makes use of certain invariance properties of the governing equations. The transformation

$$f(\eta) = A^{-1/4}F(\mu), \quad \theta(\eta) = A^{-1}\vartheta(\mu), \quad \mu = A^{-1/4}\eta \quad (26)$$

leaves the equations unchanged. For F and ϑ we may employ the asymptotic expansions of equation (22), with $\tilde{a}_0$, $\tilde{b}_0$ and μ_0 instead of a_0, b_0 and η_0. We now select a *fixed* value of $\tilde{a}_0$, take $\mu_0 = 0$ and determine F and ϑ for a value of μ that is as small as possible, but which allows small truncation errors in the series (22). The value of $\tilde{b}_0$ is still free. Now we integrate backwards, using a Runge–Kutta routine, until F changes from positive to negative. The exact location where this happens is determined accurately. Next we choose $\tilde{b}_0$ in such a way that F' vanishes as well at that very location. This entails a simple one-parameter fitting procedure. The parameter A will now be chosen as the reciprocal of the value of ϑ at this particular location. In general, the location will not be at $\mu = 0$. Then, a suitable shift determines the value of μ_0. Finally, the original parameters can be expressed in terms of the auxiliary ones,

$$a_0 = A^{-\bar{\alpha}/4}\tilde{a}_0; \quad b_0 = A^{-\bar{\alpha}/4}\tilde{b}_0; \quad \eta_0 = A^{1/4}\mu_0. \quad (27)$$

Further we have

$$f''(0) = A^{-3/4}F''(0); \quad \theta'(0) = A^{-5/4}\vartheta'(0). \quad (28)$$

For selected values of the parameters σ and n we calculated some values of the constants given above. These are listed in Tables 1 and 2. Unfortunately, the method described above is less suitable for $n = 3$, a case we are particularly interested in. However, in ref. [6] an effective way of dealing with that case was described.

When $n > 3$, the asymptotic behaviour of the functions f and θ is exponential. The equations may be solved without much difficulty by the application of a simple forward integration procedure of the Runge–Kutta type. The case that is of special interest is $n = 7$. From equation (19) we see that this corresponds to $m = 0$, i.e. a slab of constant thickness. This problem is related to that treated in ref. [1] for the downward-projecting fin. In Table 3 we list some results pertinent to this case.

5. DISCUSSION

We have already remarked that the class of backward similarity solutions put forward in this paper is complementary to the classical ones [3, 4]. In those papers wall temperatures of the kind

$$T_w = T_\infty + \Delta T(x/x_0)^{\bar{n}} \quad (29)$$

were considered in the interval $x \geqslant 0$. The equations presented in refs. [3, 4] yield a solution only if $\bar{n} > -1$. Clearly, for more negative values of $\bar{n}$ the singularity at the leading edge is too strong to be physically meaningful. When the Sparrow–Gregg–Finston equations are compared with ours [equations (15) and (16)], it is seen that replacing n by $-\bar{n}$ in equations (15) and (16) does not yield full correspondence. The coefficients in the various equations conform as to the

Table 1. Values for $n = 2$ and various Prandtl numbers

σ	$f''(0)$	$\theta'(0)$	η_0	a_0	b_0
0.5	0.88556886	−0.56150892	11.699507	2.4999695	1.4420731
1.0	0.79862076	−0.69896509	10.269562	1.9281477	0.4432221
2.0	0.70983603	−0.86083381	9.916505	1.4906044	0.1223102

Table 2. Values for $\sigma = 0.73$ and various values of n

n	$f''(0)$	$\theta'(0)$	η_0	a_0	b_0
1.25	1.02405081	−0.57141158	2.402751	1.5084231	1.1909296
1.50	0.94156666	−0.59398057	3.893245	1.7324023	1.0926840
1.75	0.88303542	−0.61468386	6.316341	1.9555110	0.9443179
2.00	0.83854651	−0.63371875	10.773539	2.1690912	0.7664254
2.25	0.80312167	−0.65131397	20.612218	2.3679279	0.5741435
2.50	0.77395437	−0.66767439	50.074713	2.5479661	0.3779707

Table 3. Values for $n = 7$ and various Prandtl numbers

σ	$f''(0)$	$\theta'(0)$
0.1	0.719320	−0.425641
0.2	0.670993	−0.551389
0.5	0.597854	−0.757887
0.73	0.565555	−0.857863
1	0.538216	−0.948140
2	0.477794	−1.172007
5	0.400877	−1.528884
10	0.347164	−1.854066

absolute value, but a difference in sign remains. This shows that our system is basically different from the one presented in refs. [3, 4]. Whereas the older system applies when $\bar{n} > -1$, the present one can be used only when $\bar{n} < -1$ (i.e. $n > 1$).

The heat flux q at the wall of the slab can be obtained from equations (13) and (14)

$$q = -k\frac{\partial T}{\partial y}\bigg|_{y=+0} = k\Delta T Gr_x^{1/4} \times \left(1-\frac{x}{x_0}\right)^{-n-1} x_0^{-1}[-\theta'(0)]. \quad (30)$$

It will be interesting to compare this result with a heat-flux result given in ref. [1]. Since the analysis in ref. [1] applies to fins of finite length, a comparison is meaningful only when the fin considered [1] loses most of its heat in a region well above its lowest edge. In that case the lengthening of the fin will not influence the heat-transfer characteristics in the salient region. We shall carry out the comparison with the result presented in Fig. 10 of ref. [1]. In that particular figure the heat-transfer number N_{cc} that was introduced by Sparrow and Acharya was reasonably large. Although our analysis strictly applies to the case of infinite fins only, i.e. for $N_{cc} \to \infty$, it will be interesting to see how closely it approximates the case $N_{cc} = 6$ studied in ref. [1]. From that paper and our equation (30), introducing an obvious change of notation we find

$$H = \frac{qL}{k\Delta T Gr_{x_0-L}^{1/4}} = \frac{1}{6}\frac{kL^2}{k_s\delta_0 x_0} \times \left(1-\frac{x}{x_0}\right)^{-n-1} Gr_x^{1/4}[-\theta'(0)], \quad (31)$$

where L is the length of the fin, and our definitions of Gr_x and x have been taken. Using the value of δ_0 given by equation (20) we obtain further

$$H = \frac{n(5n-3)}{24}\frac{L^2}{x_0^2}\left(1-\frac{x}{x_0}\right)^{-(5n+1)/4}. \quad (32)$$

As Sparrow and Acharya considered fins of uniform thickness, we have $m = 0$, i.e. $n = 7$ [equations (1) and (19)]. From equation (1) we see that the parameter x_0 does not have a precise meaning. For every other value of m it is the shape of the slab that determines x_0, but this is not so for $m = 0$. It is possible, however, to fix x_0 in terms of L by comparison of the heat input at the top of the slab, equation (21), with that of the fin [equation (19) of ref. [1], where the definition of Q yields a value that is twice that used in the present paper]. From ref. [1], we find for this heat input

$$\frac{k_s\Delta T}{L} = \frac{N_{cc}R(N_{cc})}{2}, \quad (33)$$

where $R(N_{cc})$ is given by the solid line of Fig. 2 of ref. [1]. This should be equal to the value obtainable from equation (11), whence (taking $N_{cc} = 6$)

$$\frac{7}{x_0} = \frac{3R(6)}{L}. \quad (34)$$

Since $R(6) \sim 0.575$, we may finally write equation (32) as follows:

$$H = 0.567\left[1-0.246\frac{x}{L}\right]^{-9}. \quad (35)$$

For $-L \leqslant x \leqslant 0$ this curve is depicted in Fig. 2, together with the corresponding one presented in ref. [1]. It is seen that the correspondence is excellent in the upper half of the fin. In the lower regions the two curves grow apart due to the nearness of the leading edge, which had to be expected.

Using the same value of x_0 defined by equation (34) we have also [see equation (11)]

$$\frac{T_w(0)-T_w(x)}{T_w(0)-T_\infty} = \left[1-0.246\frac{x}{L}\right]^{-7}. \quad (36)$$

Figure 3 shows a comparison of this result with that of Fig. 11 of ref. [1]. It is seen that the case $N_{cc} = \infty$ is a good approximation to that of $N_{cc} = 6$. It should be emphasized that a true comparison of our results with those of ref. [1] would only have been possible, if the technique put forward in ref. [1] had been carried out for a really large value of N_{cc}, say 100. Indeed, our choice of x_0, which led to equations (35) and (36), was based on the equalization of the cases $N_{cc} = 6$ and $N_{cc} = \infty$ on the basis of a corresponding total heat

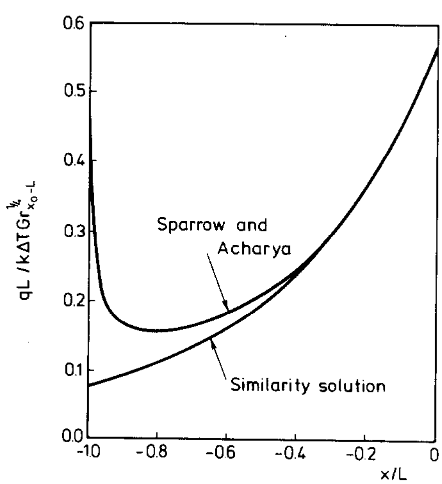

FIG. 2. Dimensionless wall heat flux along the fin. Sparrow and Acharya refer to $N_{cc} = 6$.

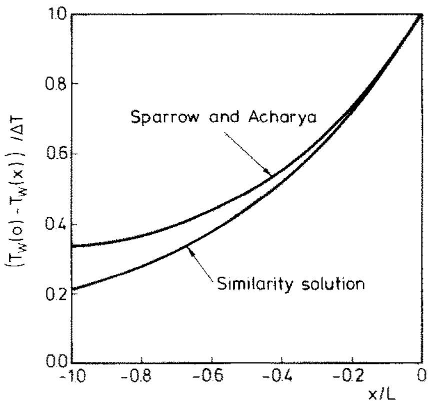

FIG. 3. Dimensionless temperature distribution along the fin. Sparrow and Acharya refer to $N_{cc} = 6$.

flux. Despite this uncertainty, the agreement between the two analyses seems to be fairly convincing.

It is interesting to note that the backward boundary layers considered in this paper become thinner as we move in the downstream direction. Indeed, from equation (14) we find that

$$y_{\text{edge}} \sim c\left(1 - \frac{x}{x_0}\right)^{(n+1)/4} \tag{37}$$

where c is a constant. For $1 < n < 3$ the boundary layer is convex with respect to the wall, whereas it is concave when $n > 3$. The consequence of this behaviour for the streamlines was discussed in ref. [7] for a similar type of problem. Although at first sight it may seem unphysical for a boundary layer to become thinner downstream, this kind of behaviour has been noted before. The Schlieren photographs presented by Lock and Gunn [2] clearly show a thinning of the boundary layer near the upper reaches of the fin. Since these authors took the leading edge at the fin's lower end, they modelled their experimental results on the class of wall temperatures (29) and they invariably found values of $\bar{n}$ that were larger than unity. It can easily be checked that the Sparrow–Gregg–Finston boundary layers become thinner downstream when $\bar{n} > 1$.

Next we shall give some thought to the condition that allows us to use a uniform temperature distribution across the slab. It would seem reasonable to state that the assumption is valid when

$$\frac{T_\delta - T_w}{T_w - T_\infty} \ll 1 \quad \text{for all } x \tag{38}$$

where T_δ is the temperature at $y = -\delta$. A more stringent condition is

$$-\delta \frac{\partial T}{\partial y}\bigg|_{y=-0} (T_w - T_\infty)^{-1} = -\frac{k}{k_s} \frac{\delta \frac{\partial T}{\partial y}\Big|_{y=+0}}{T_w - T_\infty} \ll 1. \tag{39}$$

Using equations (13) and (14), this condition can be written as follows:

$$\left(\frac{k}{k_s}\right)^2 [\theta'(0)]^2 \, Gr_x^{1/2} \ll 1. \tag{40}$$

Of course, in a small region near the upper end of the slab, the heat transfer will be truly 2-dim. The length of this region is of the order of δ_0. To describe the temperature distribution there, we ought to know precisely how the heat flows into the slab at $x = 0$. This problem could be solved by the method put forward in ref. [8], where techniques of matched asymptotic expansions were employed. However, the slab being slender, this region is so small that the exercise would not seem worth the effort involved.

Since $\theta'(0)$ is of order unity, that is to say for Prandtl numbers not too different from unity, and since Gr_x ought to be much larger than unity on account of the boundary-layer assumptions, we must clearly demand that k_s should be much larger than k for equation (40) to be satisfied. At a Grashof number of 10^8, which is just within the laminar-flow regime, k/k_s ought to be at most of the order of one thousandth. This means that, if water is taken as the working fluid, the slab must necessarily consist of a metal that is a good heat conductor. Should the fluid be a gas at atmospheric pressure, a poor heat-conducting metal such as austenitic iron would suffice.

To conclude, it might be worth mentioning that the similarity solutions described in this paper may be used to set up an expansion scheme for the solution of the finite-fin problem. The solution technique would involve two separate series expansions, the one applicable near the leading edge of the fin and the second in the upper reaches. The latter series would have the new similarity solution for its leading term. The leading term of the first expansion would belong to the Sparrow–Gregg–Finston class, namely the one for $\bar{n} = 0$. It is easily understood that the present similarity solution is a good approximation along most of the fin, if the fin is much longer than the characteristic length x_0.

REFERENCES

1. E. M. Sparrow and S. Acharya, A natural convection fin with a solution-determined non-monotonically varying heat transfer coefficient, *J. Heat Transfer* **103**, 218–225 (1981).
2. G. S. H. Lock and J. C. Gunn, Laminar free convection from a downward-projecting fin, *J. Heat Transfer* **90**, 63–70 (1968).
3. E. M. Sparrow and J. L. Gregg, Similar solutions for free convection from a nonisothermal vertical plate, *Trans. ASME* **80**, 379–386 (1958).
4. M. Finston, Free convection past a vertical plate, *Z. Angew. Math. Phys.* **7**, 527–529 (1956).
5. S. Goldstein, On backward boundary layers and flow in converging passages, *J. Fluid Mech.* **21**, 33–45 (1965).
6. H. K. Kuiken, A 'backward' free-convective boundary layer, *Q. J. Mech. Appl. Math.* **39**, 397–413 (1981).
7. H. K. Kuiken, On boundary layers in fluid mechanics that decay algebraically along stretches of wall that are not vanishingly small, *IMA J. Appl. Math.* **27**, 387–405 (1981).
8. H. K. Kuiken, The cooling of low-heat-resistance cylinders by radiation, *J. Engng Math.* **13**, 97–106 (1979).

UNE CLASSE DE SOLUTIONS RETROGRADES ET AFFINES DE COUCHE LIMITE DE CONVECTION NATURELLE

Résumé—Cet article présente une classe de solutions rétrogrades et affines de couche limite de convection naturelle. On montre que ces couches limites peuvent être produites le long de plaques minces se projetant vers le bas, d'épaisseur variant convenablement et infiniment longues. On montre que ces solutions peuvent être utilisées pour décrire les écoulements de convection naturelle le long d'ailettes verticales qui ont retenu l'attention par le passé. Un résultat important est que la température le long d'une ailette d'épaisseur constante varie en proportion inverse de la septième puissance de la coordonnée longitudinale.

EINE KLASSE VON RÜCKWÄRTIGEN ÄHNLICHKEITSLÖSUNGEN FÜR DIE GRENZSCHICHT BEI FREIER KONVEKTION

Zusammenfassung—In der vorliegenden Arbeit wird eine Klasse von rückwärtigen Ähnlichkeitslösungen für die Grenzschicht bei freier Konvektion angegeben. Es wird gezeigt, daß solche Grenzschichten sich entlang dünner, abwärts gerichteter Platten von unendlicher Länge mit vorgeschriebener Änderung der Dicke entwickeln können. Es wird darauf hingewiesen, daß diese Lösungen zur Beschriebung der freien Konvektionsströmung an vertikalen Rippen, wie sie auch zuvor schon in der Literatur behandelt wurden, verwendet werden können. Ein wesentliches Ergebnis dabei ist, daß sich die Temperatur längs einer abwärts gerichteten Rippe konstanter Dicke proportional zum Reziprokwert der siebenten Potenz der verschobenen Längskoordinate ergibt.

КЛАСС АВТОМОДЕЛЬНЫХ РЕШЕНИЙ ДЛЯ СВОБОДНОКОНВЕКТИВНОГО ПОГРАНИЧНОГО СЛОЯ С ОБРАТНЫМ ТЕЧЕНИЕМ

Аннотация—Представлен класс автомодельных решений для свободноконвективного пограничного слоя с обратным течением. Показано, что такие пограничные слои могут развиваться на тонких выступающих вниз пластинах бесконечной длины с заданным изменением толщины. Отмечено, что эти решения можно использовать для описания свободноконвективного обтекания вертикальных ребер, исследованию которого были посвящены ранее опубликованные работы. В качестве важного результата установлено, что температура вдоль выступающего вниз ребра изменяется обратно пропорционально седьмой степени продольной координаты.